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PROBLEMS OF STEADY AND UNSTEADY CREEP

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Properties of the equations describing the creep of metals are investigated. The unique solvability of the stationary boundary value problem and of the nonstationary initial-boundary value problem, attainment of the stationary mode by the solution of the nonstationary equation and the convergence of the numerical solution algorithms, are all proved.

1. Statement of the problem. In the plane theory of creep of metals, the equations connecting the deformation rate tensor with the stress tensor have the form [1]

$$\xi_{xx} = \frac{1}{3} f^\circ(T, \tau)\,(2\sigma_{xx} - \sigma_{yy}) + \frac{1}{2G}\frac{\partial}{\partial \tau}\left(\sigma_{xx} - \frac{\nu}{1+\nu}(\sigma_{xx} + \sigma_{yy})\right) \qquad (1.1)$$

$$\xi_{yy} = \frac{1}{3} f^\circ(T, \tau)\,(2\sigma_{yy} - \sigma_{xx}) + \frac{1}{2G}\frac{\partial}{\partial \tau}\left(\sigma_{yy} - \frac{\nu}{1+\nu}(\sigma_{xx} - \sigma_{yy})\right)$$

$$\xi_{xy} = f^\circ(T, \tau)\,\sigma_{xy} + \frac{1}{G}\frac{\partial}{\partial \tau}\sigma_{xy}$$

$$T = (\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2)^{1/2}, \quad (x, y) \in \Omega$$

Here Ω is a bounded region on a plane, τ is time and T is the stress intensity. We assume that the function $f^\circ(T, \tau)$ monotonous and continuous in $T \in [0, \infty)$ for every τ, satisfies the inequality

$$c_1 + c_2 T^m \leqslant \frac{1}{\gamma(\tau)} f^\circ(T, \tau) \leqslant c_3 T^m + c_4$$

where the constants $c_1 \geqslant 0$ and $c_i > 0$, $i = 2, 3, 4$ and $m \geqslant 0$. In addition, a positive function $\gamma(\tau)$ continuous in $\tau \in (0, \infty)$ is such that the function $f^\circ(T, \tau) / \gamma(\tau)$ is continuous in $\tau \in [0, \infty]$ for every T and

$$\int_0^\tau \gamma(p)\,dp \to \infty, \quad \tau \to \infty$$

System (1.1) together with the equations of equilibrium (1.2) and compatibility (1.3),

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma}{\partial x} \qquad (1.2)$$

$$\frac{\partial^2 \xi_{xx}}{\partial y^2} + \frac{\partial^2 \xi_{yy}}{\partial x^2} = 2\frac{\partial^2 \xi_{xy}}{\partial x\, \partial y} \qquad (1.3)$$

represents a closed system of equations.

In Sect. 2 we consider the initial-boundary value problems for the equations (1.1) — (1.3), and discuss their properties. Let us describe briefly these problems. In the first initial-boundary value problem the displacements $u_i = \varphi_i$ are given on the boundary of the region, and the stress tensor σ_{ij} at the initial instant of time. Inhomogeneity of the boundary conditions for u_i is contained in a separate term in the left-hand side of (1.1)

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial\varphi_i}{\partial x_j} + \frac{\partial\psi_j}{\partial x_i}\right) + \xi_{ij}$$

In accordance with the Castigliano principle [2] the generalized solution of the problem obtained is represented by the functions $\sigma_{ij}(\tau)$ satisfying (1.2), for which the identity

$$\int_0^{\tau_0} d\tau \int_\Omega [\xi_{ij} - \Phi_{ij}(\sigma_{ij})]\, \sigma_{ij}{}^\circ\, dx\, dy = 0 \tag{1.4}$$

holds for any ζ_{ij} satisfying (1.2). Here $\xi_{ij} = \Phi_{ij}(\sigma_{ij})$ is an abbreviated notation describing the system (1.1). The above integral identity does not contain the components ζ_{ij}, since for any ζ_{ij} satisfying (1.3) we have

$$\int_\Omega \zeta_{ij}\sigma_{ij}{}^\circ\, dx\, dy = 0$$

The quantities σ_{ij} obtained from (1.4) and substituted into the right-hand side of (1.1), determine ξ_{ij} uniquely. The procedure is equivalent to projecting the system (1.1) — (1.3) on the subspace of solutions of (1.3), and makes possible the elimination of the tensor component ζ_{ij} from the equations under consideration.

In the second initial-boundary value problem the force applied at the boundary of the region is given. Using the Airy function we can reduce the system (1.1) — (1.3) to a quasi-linear equation, and its generalized solution can be obtained in the usual manner.

In Sect. 3 we study the behavior of the solution as $\tau \to \infty$ and prove that it tends to the solution of a stationary equation on some norm.

In Sect. 4 we investigate the approximate methods of solving these problems.

2. Initial-boundary value problems. Performing a change of variables, we write the system (1.1) in the matrix form as follows:

$$\frac{d}{dt} A\sigma + \frac{1}{3} f(t, T)\, B\sigma = \xi \tag{2.1}$$

$$t = \int_0^\tau \gamma(p)\, dp, \qquad \sigma = \begin{Vmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Vmatrix}, \qquad \xi = \begin{Vmatrix} \xi_{xx} \\ \xi_{yy} \\ \xi_{xy} \end{Vmatrix}$$

$$A = \frac{1}{G(\nu+1)} \begin{Vmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(\nu+1) \end{Vmatrix}, \qquad B = \begin{Vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{Vmatrix}$$

The function $f(t, T) \in C[0, \infty]$ in t for every T and satisfies the inequality

$$c_1 + c_2T^m \leqslant f(t, T) \leqslant c_3T^m + c_4 \tag{2.2}$$

Let us consider the following boundary and initial conditions:

$$u_1\big|_{\partial\Omega} = \varphi, \quad u_2\big|_{\partial\Omega} = \psi, \quad \sigma\big|_{t=0} = \sigma_0 \tag{2.3}$$

We assume that the functions φ and ψ can be continued into the interior of the region Ω on the class $W^2_{(m+2)/(m+1)}$.

By regarding the derivatives in a more general sense we find, that the left-hand side of the system (1.2) defines a closed operator in the space of functions with a finite norm

$$\sum_{i,j}\Big(\int_\Omega |\sigma_{ij}|^{m+2}\,dx\,dy\Big)^{1/(m+2)} \tag{2.4}$$

Let us denote by N the closed space of zeros of this operator. By definition, $\sigma \in N$ if and only if the identity

$$\int_\Omega \left[\sigma_{xx}\frac{\partial\psi_1}{\partial x}+\sigma_{xy}\frac{\partial\psi_1}{\partial y}+\sigma_{yy}\frac{\partial\psi_2}{\partial y}+\sigma_{xy}\frac{\partial\psi_2}{\partial x}\right]dx\,dy=0$$

holds for any $\psi_i \in \overset{\circ}{W}{}^{2}_{(m+2)/(m+1)}$, $i=1,\ 2$, and the integral in (2.4) is bounded. We assume that $\sigma_0 \in N$. In addition, we shall define a space of functions $\sigma(t) \in N$, $t \in [0,\ t_0]$ with the finite norm

$$\Big(\int_0^{t_0} \|\sigma(t)\|_N^{m+2}\,dt\Big)^{1/(m+2)}$$

and we denote this space by N_{t_0}

We write ξ in the form of a sum

$$\xi=\xi_0+\zeta,\ \xi_{ij}=\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j}+\frac{\partial w_j}{\partial x_i}\right),\ w_i\big|_{\partial\Omega}=0$$

where ξ_0 is constructed in terms of φ and ψ given by (2.3). In accordance with the Castigliano principle [2] we shall call the function $\sigma(t) \in N_{t_0}$ with a finite norm

$$\Big(\int_0^{t_0}\|\sigma(t)\|_N^{m+2}\,dt\Big)^{1/(m+2)}+\Big(\int_0^{t_0}\Big\|\frac{d\sigma(t)}{dt}\Big\|_{N^*}^{(m+2)/(m+1)}dt\Big)^{(m+1)/(m+2)} \tag{2.5}$$

and the function $\zeta(t)$ with a finite norm

$$\Big(\int_0^{t_0}\|\zeta_{ij}(t)\|_{L_{m+2}(\Omega)}\,dt\Big)^{1/(m+2)} \tag{2.6}$$

such that the identity

$$\int_0^{t_0}dt\int\Big(\frac{d}{dt}A\sigma+\frac{1}{3}f(t,T)B\sigma,\ \eta\Big)\,dx\,dy=\int_0^{t_0}dt\int_\Omega(\xi_0,\ \eta)\,dx\,dy \tag{2.7}$$

holds for any $\eta_{ij}(t) \in N_{t_0}$, the generalized solution of the problem (1.1) — (1.3) and (2.3), on the interval $[0,\ t_0]$.

Functions with a finite norm (2.7) are continuous in the Hilbert space H which is the kernel of the operator (1.2) given on the space of square summable functions σ_{ij} [3]. For the trio of spaces N, H and N^* (where N^* is a space conjugate to N) the inclusion $N \subset H \subset N^*$ holds, and is dense. The identity (2.7) leads to the equation

$$\frac{d}{dt}A\sigma(t)+K(t)\sigma(t)=\xi_0,\quad \sigma\big|_{t=0}=\sigma_0 \tag{2.8}$$

where the operator $K(t)$ acts from N into N^*.

Theorem 1. Operator $K(t)$ is bounded and continuous for every t, and is strongly monotonous uniformly in $t \in [0,\ t_0]$.

Proof. The continuity and boundedness follow from the properties of the function $f(t,\ T)$ (see [4]). We shall prove the strong monotony. We have

$$\langle K\sigma_1-K\sigma_2,\ \sigma_1-\sigma_2\rangle=\int_\Omega\Big[\frac{1}{3}(f(t,T_1)(2\sigma^1_{xx}-\sigma^1_{yy})-$$

$$f(t, T_2)(2\sigma^2_{xx} - \sigma^2_{yy}))(\sigma^1_{xx} - \sigma^2_{xx}) + \frac{1}{3}(f(t, T_1)(2\sigma^1_{yy} - \sigma^1_{xx}) -$$
$$f(t, T_2)(2\sigma^2_{yy} - \sigma^2_{xx}))(\sigma^1_{yy} - \sigma^2_{yy}) + 2(f(t, T_1)\sigma^1_{xy} -$$
$$f(t, T_2)\sigma^2_{xy})(\sigma^1_{xy} - \sigma^2_{xy})]\,dx\,dy = \int_\Omega \Big[\frac{1}{3}(f(t, T_1) -$$
$$f(t, T_2)(T_1 - T_2) + \frac{1}{3} = f(t, T_1) + f(t, T_2))(|\sigma^1_{xx} - \sigma^2_{xx}|^2 +$$
$$|\sigma^1_{yy} - \sigma^2_{yy}|^2 + 3|\sigma^1_{xy} - \sigma^2_{xy}|^2 - (\sigma^1_{xx} - \sigma^2_{xx})(\sigma^1_{yy} - \sigma^2_{yy}))]\,dx\,dy$$

Using the fact that $f(t, T)$ is monotonous in T and the inequality

$$\frac{1}{3}(f(t, T_1) + f(t, T_2)) \geqslant \frac{c_2}{6(\sqrt{2})^m}[|\sigma^1_{xx} - \sigma^2_{xy}|^2 + |\sigma^1_{yy} - \sigma^2_{yy}|^2 + |\sigma^1_{xy} - \sigma^2_{xy}|^2]^{m/2}$$

we obtain

$$\langle K\sigma^1 - K\sigma^2, \sigma^1 - \sigma^2\rangle \geqslant \frac{c_2}{12(\sqrt{2})^m}\|\sigma^{-1} - \sigma^2\|_N^{m+2} \tag{2.9}$$

From the theorem and the positiveness of the matrix A it follows [3] that Eq. (2.8) has a unique solution.

Let us consider another boundary value problem in which a force is given at the boundary of the region Ω. Expressing the stress tensor in terms of the Airy function and using the equation of compatibility, we arrive at the following problem:

$$\frac{\partial}{\partial t}\Delta^2 F + 2G(1 + \nu)\Big\{\frac{1}{6}\frac{\partial^2}{\partial x^2}[f(t, T)(2F_{xx} - F_{yy})] + \tag{2.10}$$
$$\frac{1}{6}\frac{\partial^2}{\partial y^2}[f(t, T)(2F_{yy} - F_{xx})] + \frac{\partial^2}{\partial x\,\partial y}[f(t, T)F_{xy}]\Big\} = 0$$
$$T^2 = F^2_{xx} + F^2_{yy} - F_{xx}F_{yy} + 3F^2_{xy}, \quad F\big|_{t=0} = F(0)$$
$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}$$

If the region is singly connected, then the boundary conditions can be reduced to the form

$$F\,|_{\partial\Omega} = \psi_1, \quad \partial E\,/\,\partial n\,|_{\partial\Omega} = \psi_2$$

A similar stationary problem was studied in [5], but under the assumptions made there about the function $f(T)$ ($Tf(T)$ monotonous in T), the problem is only reduced to a strictly monotonous operator; this is insufficient for investigating the behavior of the solution of the problem (2.10) as $t \to \infty$ and for constructing algorithms for determining the approximate solution.

Let us write F in the form

$$F = \varphi + F_0, \quad F_0 \in W^2_{m+2}$$

where φ satisfies a homogeneous boundary condition, and use the spaces $\overset{0}{W}{}^2_{m+2}$ and $\overset{0}{W}{}^2_2$ as N and H respectively. Then, as in the previous case, the problem (2.10) reduces to

$$\frac{d\varphi}{dt} + Q(t)\varphi = 0, \quad \varphi\big|_{t=0} = \varphi_0 \tag{2.11}$$

Theorem 2. Operator $Q(t)$ which maps N in N^* is bounded, continuous and strongly monotonous for every $t \in [0, t_0]$.

The proof is similar to that of Theorem 1, but the inequality (2.9) is replaced by

$$\langle Q\varphi_1 - Q\varphi_2,\ \varphi_1 - \varphi_2\rangle \geqslant \frac{c_2}{12(\sqrt{2})^m}\|\varphi_1 - \varphi_2\|_N^{m+2} \tag{2.12}$$

From Theorem 2 it follows [3] that Eq. (2.11) has a unique solution.

Let $f_0(T) = \lim f(t, T)$ as $t \to +\infty$ and let the operators K_0 and Q_0 be generated by the function $f_0(T)$. Then by virtue of (2.9) and (2.12), the problems

$$K_0\sigma = 0, \quad Q_0\varphi = 0 \tag{2.13}$$

have unique solutions. We shall consider the second problem of (2.13) in more detail.

Theorem 3. Let the constant c_1 in (2.4) be positive, $m \geqslant 1$ and

$$f_0(T) \in C^1[0, \infty),\ F_0 \in C^4(\overline{\Omega})$$

$$c_2' T^{m-1} \leqslant \frac{df_0(T)}{dT} \leqslant c_3' T^{m-1}, \quad c_3' \geqslant c_2' > 0$$

Then the operator Q_0 is strictly elliptic.

Proof. We introduce the following notation:

$$\varphi_{xx} = \xi_1,\quad \varphi_{yy} = \xi_2,\quad \varphi_{xy} = \xi_3,\quad \varphi_{yx} = \xi_4$$

$$F_{0xx} = \eta_1,\quad F_{0yy} = \eta_2,\quad F_{0xy} = \eta_3,\quad F_{0yx} = \eta_4$$

$$\xi_i + \eta_i = \zeta_i,\quad T^2 = \zeta_1^2 + \zeta_2^2 + {}^3/_2\,\zeta_3^2 + {}^3/_2\,\zeta_4^2 - \zeta_1\zeta_2$$

$$D_1 = {}^1/_6\, f_0(T)(2\zeta_1 - \zeta_2),\quad D_2 = {}^1/_6\, f_0(T)(2\zeta_2 - \zeta_1)$$

$$D_3 = {}^1/_2\, f_0(T)\,\zeta_3,\quad D_4 = {}^1/_2\, f_0(T)\,\zeta_4$$

Consider the matrix $D = \{\partial D_i / \partial \xi_j\}$ and the corresponding linear operator. It is sufficient to show that (see [5]) the inequality

$$(D\alpha, \alpha) \geqslant c\left(a + \sum_{i+1}^{4} |\xi_i|\right)^m \|\alpha\|^2, \quad c, a > 0$$

holds for any vector $\alpha \in R^4$. We can show by direct computation that in the conditions of the theorem all principal minors of the matrix D are positive. Therefore the form $(D\alpha, \alpha)$ can be reduced by triangular transformation to a diagonal form with positive coefficients and, by virtue of the law of inertia, all eigenvalues of the symmetric matrix D are positive. It can be shown by computation that

$$\det D \geqslant \frac{f_0^4(T)}{48}, \quad \|D\| \leqslant 2^4\left[f_0(T) + \frac{df_0(T)}{dT}T\right]$$

Using the estimate (see [6])

$$1 / |\lambda_{\min}| = \|D^{-1}\| \leqslant \|D\|^3 / |\det D|$$

we obtain

$$\lambda_{\min} \geqslant c\left(a + \sum_{i=1}^{4} |\xi_i|\right)^m$$

which proves the theorem. From this it follows that when the boundary $\partial\Omega$ is sufficiently smooth, then the solution of the problem $Q_0\varphi = 0$ belongs to the space $C^{2,\delta}$ for some value of δ [5].

3. Behavior of the solution as $t \to \infty$.

Lemma 1. Under the assumptions made in Sect. 1 concerning the dependence of $f(t, T)$ on t, the operators $K(t)$ and $Q(t)$ are strongly continuous in $t \in [0, \infty]$.

Proof. For any t_1 and $t_2 \in [0, \infty]$, φ and $\psi \in N$, we have

$$|\langle Q(t_1)\varphi - Q(t_2)\varphi, \psi\rangle| \leqslant \|\psi\|_N \left[\int_\Omega \left\{\left[f(t_1,T) - f(t_2,T)\right] \times \sum_{|\alpha|=2} |D^\alpha F|\right\}^{1/n}\right]^n, \quad n = \frac{m+1}{m+2}$$

The integrand tends to zero as $t_1 \to t_2$ and, by virtue of the inequality (2.2) has a summable majorant. Then according to the Lebegue theorem

$$\| Q(t_1)\varphi - Q(t_2)\varphi \|_{N^*} = \sup \|\psi\|_N = 1\, |\langle Q(t_1)\varphi - Q(t_2)\varphi, \psi\rangle| \to 0$$

For the operator $K(t)$ the proof is analogous.

Theorem 4. When $t \to \infty$, the solutions of (2.8) and (2.11) tend in the space H to solutions of the first and second problem of (2.13), respectively.

Proof. From the relation

$$\frac{d}{dt}(\varphi(t) - \varphi) + Q(t)\varphi(t) - Q(t)\varphi + Q(t)\varphi - Q_0\varphi = 0$$

where $Q_0\varphi = 0$ and the formula (2.12) we obtain, after multiplying by $\varphi(t) - \varphi$ and integrating in t from t_1 to t_2

$$\frac{1}{2}y(t_2) - \frac{1}{2}y(t_1) \leqslant - \frac{c_2}{12(\sqrt{2})^m}\int_{t_1}^{t_2} \|\varphi(t) - \varphi\|_N^{m+2}\, dt + \int_{t_1}^{t_2} \|\varphi(t) - \varphi\|_N \, \|Q(t)\varphi - Q_0\varphi\|_{N^*} dt, \quad y(t) = \|\varphi(t) - \varphi\|_H^2 \tag{3.1}$$

From (3.1), the Hölder inequality in ε and Lemma 1, follows

$$y(t_2) - y(t_1) \leqslant -a\int_{t_1}^{t_2} (y(t)^{m/2+1} - \gamma(t))\, dt \tag{3.2}$$

$$a > 0, \quad \gamma(t) \in C[0, \infty], \quad \gamma(t) \geqslant 0$$

From (3.2) we find that $y(t)$ cannot be strictly larger than any ε when $t \in [0, \infty]$. If $y(t)$ does not tend to zero as $t \to \infty$, then a sequence of intervals $[t_1^i, t_2^i]$ can be found such that $\varepsilon = y(t_1^i) = y(t_2^i)$, $y(t) > \varepsilon$ for $t \in (t_1^i, t_2^i)$, with i assuming the values ranging either from 0 to ∞, or from 0 to k. In the first case $t_{1,2}^i \to \infty$ as $i \to \infty$, and in the second case $t_1^k < \infty$, $t_2^k = \infty$. The assumption of the existence of such intervals contradicts the inequality (3.2). In fact, setting $t_1 = t_1^i$ and $t_2 \in (t_1^i, t_2^i)$ for sufficiently large (or the last value) of i, from (3.2), by virtue of the fact that $\gamma(t) \to 0$ as $t \to \infty$, we obtain the following contradiction

$$\varepsilon < y(t_2) < y(t_1) < \varepsilon$$

When Q is independent of t, from (3.2) it follows that $y(t) \to 0$ monotonously. For the problem (2.8) the proof is similar.

4. Approximate methods of solution and their convergence.

All approximate methods can be based on a single scheme. For this reason we shall consider, as an example, the method of finite elements, which finds at present most applications in the theory of elasticity and plasticity. We construct a regular partition of the region into a manifold Ω_h of finite elements. Under the regular partition we understand such partition in which the neighboring elements have either a common vertex or a common side, and can be mapped onto an isosceles right-angled triangle by means of a nonsingular transformation.

Let piecewise polynomial functions $\varphi_{ij} \in N$ (i, j is the node number) different from zero only in the elements containing the given node i, j be specified on Ω_h. We shall seek the p-th approximation in the form

$$\varphi^{(p)} = \sum_{i,j\in\Omega_h} c_{ij}^p \varphi_{ij} \tag{4.1}$$

where the coefficients $c_{ij}{}^p$ are found from the system of equations

$$\langle Q_0\varphi^{(p)}, \ \varphi_{ij}\rangle = 0, \quad i, j \in \Omega_h \tag{4.2}$$

By (2.12), the system (4.2) has a unique solution and $\| \varphi^{(p)} \|_N \leqslant c\ (p)$.

Let us substitute $\varphi_1 = \varphi$ and $\varphi_2 = \varphi^{(p)}$ into the inequality (2.12). Then for any coefficients d_{ij} we obtain

$$\left\langle Q_0\varphi^{(p)}, \sum_{i,j\subset\Omega_h} d_{ij}\varphi_{ij} - \varphi \right\rangle \geqslant \frac{c_1}{12\,(\sqrt{2}\,)^m} \|\varphi^{(p)} - \varphi\|_N^{m+2}$$

from which by virtue of the boundedness of Q_0, follows

$$\left\| \varphi - \sum_{i,j} d_{ij}\varphi_{ij} \right\|_N \geqslant c \, \|\varphi\|^{(p)} - \varphi\|_N^{m+2}$$

The convergence of $\varphi^{(p)}$ to φ follows from the fact that φ can be approximated by the supporting functions. The rate of convergence depends on the choice of functions φ_{ij} and on the smoothness of the solution φ (for more detail see [7]).

Let us now construct the algorithm from the solution of (4.2). Since

$$\sum_{|\alpha|=2} | D^\alpha \varphi^{(p)} | \leqslant c$$

$\varphi^{(p)}$ will still remain a solution of (4.2) provided that $f_0(T)$ in this system is replaced by the truncated function

$$\bar{f}_0(T) = \begin{cases} f_0(T), & T \leqslant \bar{T} = 5 \sup \sum\limits_{|\alpha|=2} | D^\alpha (F_0 + \varphi^{(p)}) | \\ f_0(\bar{T}), & T \geqslant \bar{T} \end{cases} \tag{4.3}$$

as the monotonous character of $\bar{f}_0(T)$ guarantees that the modified system has a unique solution and $\varphi^{(p)}$ satisfies this automatically. We shall denote the operator corresponding to the truncated function by $\bar{Q}_0$, assume that $c_{ij}{}^p = c_{ij}(t)$ and $c_{ij}{}^p(0) = 0$, and consider the equation

$$d\varphi^{(p)}(t) \,/\, dt + \bar{Q}_0 \, \varphi^{(p)}(t) = 0 \tag{4.4}$$

From the results of Sect. 3 it follows that $\varphi_{(t)}^{(p)}$ tends monotonously to $\varphi^{(p)}$ in H as $t \to \infty$. It is therefore sufficient to be able to solve (4.4) on any interval $[0, t_0]$. We shall solve Eq. (4.4) according to the explicit scheme

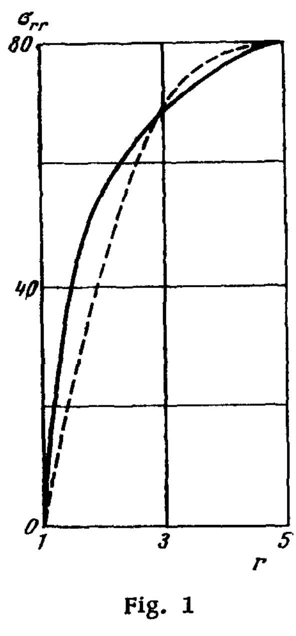

Fig. 1

$$\varphi^{(p)}((n+1)\tau) - \varphi^{(p)}(n\tau) = \tau \bar{Q}_0 \varphi^{(p)}_{(n\tau)} \tag{4.5}$$

After scalar multiplication of (4.5) by $\varphi^{(p)}_{(n+1)\tau}$ in H, we apply the Hölder inequality in ε to obtain the estimate

$$\|\varphi^{(p)}((n+1)\tau)\|_H \leqslant (1 + c\tau)\|\varphi^{(p)}_{(n\tau)}\|_H \tag{4.6}$$

from which the boundedness of $\|\varphi^{(p)}_{(n\tau)}\|$ at any N and $n = 0, \ldots, N$ follows. The estimates (4.6) and the boundedness of Q_0 in H are together sufficient to show that $\varphi^{(p)}_{(n\tau)}$ in H converges weakly to the solution of (4.4). The same scheme can be used to solve the nonstationary problem (2.10) under the additional assumption of smoothness needed for performing the truncation (4.2).

Below we give the results of the numerical computations. In the first case in which Ω is an annulus with a free inner boundary (natural boundary condition) and an axially symmetric load is applied at the outer boundary, Fig. 1 shows the dependence of σ_{rr} on r for an elastic state (solid line) and for a steady creep (dashed line) for $f(T) = T^{2.6}$. In the second case we have a square region $(x, y) \in [-20, 20] \times [-20, 20]$ with such boundary conditions, that the Airy function of elastic state has the form $F_0 - 0.1(60x^2 - x^3 + 60y^2 - y^3)$.

Table 1

y \ x	0	4	8	12	16	20	y \ x	0	4	8	12	16	20
20	1625	1702	1942	2302	2724	3207	8	341	421	672	—	—	—
16	1104	1231	1525	1823	2314	—	4	35	182	—	—	—	—
12	702	783	1004	1417	—	—	0	18	—	—	—	—	—

Table 1 gives the values of the Airy function for $y \geqslant x \geqslant 0$. In the region $x \geqslant y \geqslant 0$ the results are symmetric about the diagonal of the table. The Airy function is distributed in parity over the whole region.

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